Geometry of Harmony and Modes in Vaughan Williams' *Romanza*

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This is a preliminary report on the work of our summer 2013 REU project, GEOMETRY OF MUSICAL MODES, funded by the UW-Eau Claire Office of Research and Sponsored Programs using differential tuition. We investigated how two geometric methods, the *Tonnetz* and The Table of Diatonic Relations, are useful in analyzing some music by Vaughan Williams.

1.1 Introduction

The third movement of the *Symphony No. 5 in D-Major* (1938–43) by Ralph Vaughan Williams, the *Romanza*, is one of his most popular pieces. It exhibits many of the features that are common to his music. He makes extensive use of modal melodies, often related to English folk tunes, as well as triads and seventh chords for harmony, but *without* establishing a clear tonal center (tonic). This radically distinguishes his music from the serialism used by Schoenberg, Berg, and Webern, during the same time period, which avoids establishing a tonic by eschewing any of the tonal structures created previously in Western music. We shall see that the *Tonnetz* provides insight into the logic of the harmony used in *Romanza*. We shall also introduce some ideas that have proved useful in analyzing modal aspects of Vaughan Williams' music, but have not been used previously to study *Romanza*.

One difficulty that our discussion will pose for readers is that, due to copyright restrictions, the score for *Romanza* is not public domain in the U.S. or the E.U. Consequently, we do not make any quotations from the score. If you live in Canada, however, the score is public domain and can be downloaded from IMSLP.[1] A number of quotations that we will refer to, however, can be found in the book by Elliott Schwartz (1964) cited in the references. As a substitute for score quotations, you can follow changes in pitch classes and various chords by accessing this web page:

$$\text{people.uwec.edu/walkerjs/mathematicsandmusic/Nav/RA.htm} \tag{1}$$

It contains video animations of *Romanza* displayed on the *Tonnetz*.

1.2 Geometry of the Harmony in *Romanza*

Romanza begins with the following sequence of triadic chords:

$$\mathrm{C} \to \mathrm{A} \to \mathrm{g} \to \mathrm{A} \to \mathrm{g} \to \mathrm{A} \to \mathrm{C} \to \mathrm{A} \to \mathrm{g} \to \mathrm{A} \to \mathrm{g}. \tag{2}$$

The first progression, $\mathrm{C} \to \mathrm{A}$, is a *Tonnetz* transformation: $\mathrm{C} \xrightarrow{\mathcal{T}} \mathrm{A}$. It is an example of a chromatic mediant transformation, as described in Remark 7.4.1 on pp. 234–235 of Walker and Don (2013). In this transformation, C acts as a leading tone for $\mathrm{C}^\sharp$. The progression, $\mathrm{A} \to \mathrm{g}$, however, is not a single *Tonnetz* transformation. Nevertheless, we shall see that it has a "factorized form" related to mirror symmetries of the *Tonnetz*. It is interesting to note that the first seven chords in (2) form a palindrome:

$$\mathrm{C} \to \mathrm{A} \to \mathrm{g} \to \mathrm{A} \to \mathrm{g} \to \mathrm{A} \to \mathrm{C}.$$

All of the chords in (2) belong to the key of D-minor,[2] the parallel minor to the D-major key for the symphony. We have $\mathrm{D} \xrightarrow{\mathcal{P}} \mathrm{d}$ as a relation between these keys. In roman numeral notation, for the key of

[1] Use search topic: Vaughan Williams Symphony No. 5.

[2] If we make use of both the natural minor and harmonic minor scales, c.f. Figures 3.11 and 3.12 on pp. 63–64 of Walker and Don (2013).

D-minor, we can write the chords in (2) as follows:

$$\mathbf{VII} \to \mathbf{V} \to \mathbf{iv} \to \mathbf{V} \to \mathbf{iv} \to \mathbf{V} \to \mathbf{VII} \to \mathbf{V} \to \mathbf{iv} \to \mathbf{V} \to \mathbf{iv}. \tag{3}$$

The progressions $\mathbf{V} \to \mathbf{iv}$ and $\mathbf{iv} \to \mathbf{V}$ are covered by Tymoczko's model for chord progressions in a minor key, but not the progression $\mathbf{VII} \to \mathbf{V}$.[3] In contrast, Kostka and Payne's model only covers the progression $\mathbf{iv} \to \mathbf{V}$.[4] While these chords are consistent with a key of D-minor, there is *no emphasis* on the pitch class $\mathbf{D}$, nor does the minor chord d ever occur. Consequently, the tonic for D-minor is not established during these opening measures. This is one example of the avoidance of a tonic that we pointed out above. Schwartz (1964), however, refers to the opening as A-major (modal). The multiple appearances of the A-major chord, and the multiple appearances of the chromatic mediant transformations $\mathrm{C} \to \mathrm{A}$ and $\mathrm{A} \to \mathrm{C}$, provide strong evidence for that.

1.3 Factorizations of Progressions

There are several chord progressions in *Romanza* which can be viewed mathematically as *factorizations* on the *Tonnetz*, i.e., as splitting into two successive transformations which have clear geometric interpretations on the *Tonnetz*. We will give three examples of these factorizations. In the first two examples, the factorization is probably best viewed as a mathematical underpinning for the surface musical structure. In the third example, however, the factorization provides a more definite musical explanation.

Example 1. In (2), the progressions $\mathrm{A} \to \mathrm{g}$ and $\mathrm{g} \to \mathrm{A}$ appear on the *Tonnetz* as shown on the left of Figure 1. We have also graphed all of the moves for the first twelve chords in *Romanza*:

$$\mathrm{C} \to \mathrm{A} \to \mathrm{g} \to \mathrm{A} \to \mathrm{g} \to \mathrm{A} \to \mathrm{C} \to \mathrm{A} \to \mathrm{g} \to \mathrm{A} \to \mathrm{g} \to \mathrm{e}^7. \tag{4}$$

The last chord e^7 is viewed as an embellishment of the chord e when we graph it on the *Tonnetz*.

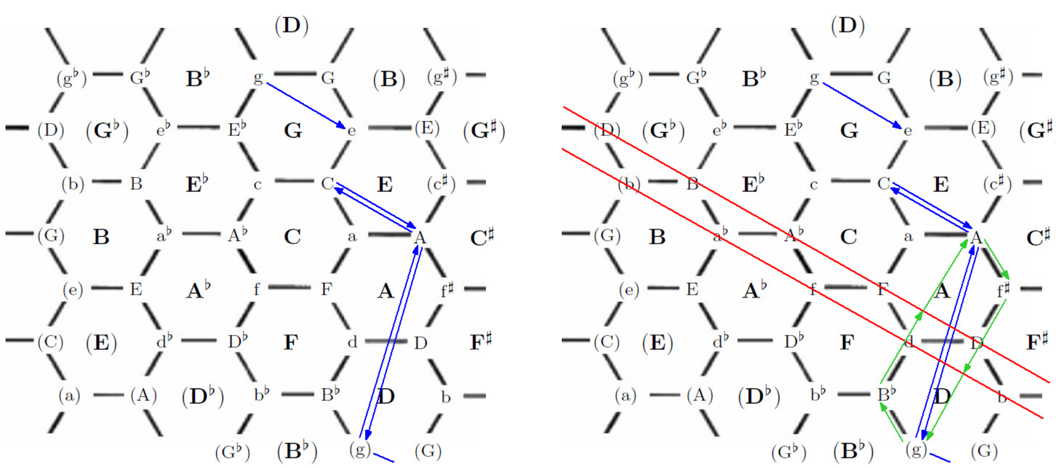

Figure 1. Left: Moves on the *Tonnetz* corresponding to first 12 chords in *Romanza*, as shown in (4). Right: Factorization of the progressions $\mathrm{A} \to \mathrm{g}$ and $\mathrm{g} \to \mathrm{A}$, using geometric operations on the *Tonnetz*.

The factorizations of $\mathrm{A} \to \mathrm{g}$ and $\mathrm{g} \to \mathrm{A}$ are shown on the right of Figure 1. First, we consider the factorization of $\mathrm{A} \to \mathrm{g}$. This is indicated in the figure as the neo-Riemannian chord transformation $\mathrm{A} \xrightarrow{\mathcal{R}} \mathrm{f}^\sharp$, followed by the transposition $\mathrm{f}^\sharp \xrightarrow{\mathsf{T}_1} \mathrm{g}$. The transformation $\mathrm{A} \xrightarrow{\mathcal{R}} \mathrm{f}^\sharp$ can be viewed as a reflection through

[3] Tymoczko (2011), p. 229. See also Figure 7.20 on p. 236 of Walker and Don (2013).
[4] Kostka and Payne (2009), p. 113. See also Figure 3.13 on p. 64 of Walker and Don (2013).

a mirror passing through the midpoint of the edge connecting the chords A and $\mathrm{f}^\sharp$, and passing through the center point of the hexagon $\mathbf{A}$. [Compare a similar mirror shown on one of the hexagons in Figure A.11 on p. 288 of Walker and Don (2013).] The transposition $\mathrm{f}^\sharp \xrightarrow{\mathsf{T}_1} \mathrm{g}$ can also be viewed as a reflection through a mirror. This is indicated on the right of Figure 1 by a mirror line drawn through the vertices for the minor chords b and d (and continuing through the other minor chords, f, $\mathrm{a}^\flat$, and b again).

Second, we consider the factorization of $\mathrm{g} \to \mathrm{A}$. This is indicated in the figure as the neo-Riemannian chord transformation $\mathrm{g} \xrightarrow{\mathcal{R}} \mathrm{B}^\flat$, followed by the transposition $\mathrm{B}^\flat \xrightarrow{\mathsf{T}_{-1}} \mathrm{A}$. The transformation $\mathrm{g} \xrightarrow{\mathcal{R}} \mathrm{B}^\flat$ can be viewed as a reflection through a mirror passing through the midpoint of the edge connecting the chords g and $\mathrm{B}^\flat$, and passing through the center point of the hexagon $\mathbf{D}$. The transposition $\mathrm{B}^\flat \xrightarrow{\mathsf{T}_{-1}} \mathrm{A}$ can also be viewed as a reflection through a mirror. This is indicated on the right of Figure 1 by a mirror line drawn through the vertices for the major chords D and F (and continuing through the other major chords, $\mathrm{A}^\flat$, B, and D again).

Remark 1. The factorizations described in Example 1 are quite satisfying mathematically. They geometrically interpret the fundamental transpositions, T_1 and T_{-1}, as reflections on the *Tonnetz*. Moreover, a mirror line for transposing a minor chord passes through minor chords, while a mirror line for transposing a major chord passes through major chords. However, these mathematical satisfactions beg the question as to whether there is a musical interpretation of these factorizations. Here is one possible way to interpret them. Beginning with the second chord C there is a prominent melody played by an English horn (see Ex. 2 on p. 98 of Schwartz, 1964). For reasons explained below, this melody is in E-Phrygian mode. However, during the second chord progression of the form $\mathrm{A} \to \mathrm{g}$ harmonizing this melody in measures 10 and 11, there occurs a tone of $\mathrm{F}^\sharp$ right at the end of measure 10. This $\mathrm{F}^\sharp$ is not on the scale for E-Phrygian, since that scale has no sharps or flats. We have seen, however, that $\mathrm{A} \to \mathrm{g}$ factors as $\mathrm{A} \xrightarrow{\mathcal{R}} \mathrm{f}^\sharp \xrightarrow{\mathsf{T}_1} \mathrm{g}$. The pitch $\mathrm{F}^\sharp$ can be viewed as a surface manifestation of the middle chord $\mathrm{f}^\sharp$ in this factorization. Similarly, there are F notes in the melody in measure 9. They can be viewed as surface manifestations of the factorization $\mathrm{g} \xrightarrow{\mathcal{R}} \mathrm{B}^\flat \xrightarrow{\mathsf{T}_{-1}} \mathrm{A}$, which is the chord progression from the harmony in measure 9 to measure 10. Although this factorization explanation is not absolutely necessary for this example,[5] we shall find it much more compelling for the transposed repetition of this passage described in Example 3.

Example 2. A second example of a factorization occurs in the chord progression:

$$\mathrm{g} \to \mathrm{e}^7 \to \mathrm{F}^{\mathrm{sus\ G}} \to \mathrm{F} \to \mathrm{F}^7. \tag{5}$$

which extends the progression shown in (4). The chord $\mathrm{F}^{\mathrm{sus\ G}}$ is a chord with pitch classes **F**, **G**, **A**, and **C**. The pitch class **G** is a "suspension" as it resolves into the pitch class **F** within the next chord F. On the left of Figure 2, we show how this progression looks on the *Tonnetz*, if we treat each of the chords with root **F** as embellishments of the triadic chord F.

The movement from e to F on the *Tonnetz* can be expressed in the factored form

$$\mathrm{e} \xrightarrow{\mathcal{L}} \mathrm{C} \xrightarrow{\mathsf{T}_{-7}} \mathrm{F}. \tag{6}$$

We illustrate this on the right of Figure 2. The neo-Riemannian transformation $\mathrm{e} \xrightarrow{\mathcal{L}} \mathrm{C}$ can be viewed as a reflection through a mirror passing through the midpoint of the edge connecting the chords e and C, and passing through the center point of the hexagon $\mathbf{E}$. The transposition $\mathrm{C} \xrightarrow{\mathsf{T}_{-7}} \mathrm{F}$ can be viewed as a reflection through

[5] There are two other explanations for the appearance of the $\mathrm{F}^\sharp$ note. One explanation is that $\mathrm{F}^\sharp$ is an intermediate chromatic tone, between the E that proceeds it, one step down, and leading into the note G that follows it, one half-step up. This voice leading into G is important, as G is the root for the chord g in the harmony, and it complements our factorization explanation. A second explanation is that $\mathrm{F}^\sharp$ represents a hint of the E-Aeolian mode, which uses a scale containing $\mathrm{F}^\sharp$, so that the entire melody is a merging of the E-Phrygian and E-Aeolian modes. (More discussion of modes can be found in Section 1.4.)

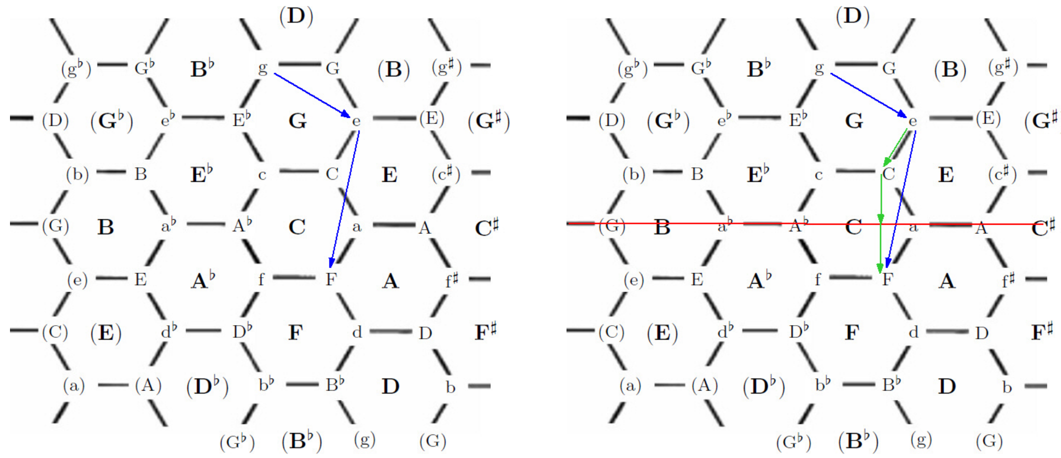

Figure 2. Left: Moves on the *Tonnetz* corresponding to the progression in (5). Right: Factorization of $\mathrm{e} \to \mathrm{F}$, using geometric operations on the *Tonnetz*.

the horizontal mirror passing through the vertices $\mathrm{A}^\flat$ and a. The factorization (6) employs the intermediary chord C. The pitch classes for this chord are a subset of the union of the pitch classes of e and F. So, although it does not appear directly in the surface structure of the music, it is not unreasonable to view it as an underlying chord within the factorization of $\mathrm{e} \to \mathrm{F}$.

Example 3. Our third example of a factorization occurs later in the *Romanza*, in measures 39 through 47. The harmony for these measures consists of the following chord progression:

$$\mathrm{G} \to \mathrm{E} \to \mathrm{d} \to \mathrm{E} \to \mathrm{G} \to \mathrm{E} \to \mathrm{d} \to \mathrm{E} \to \mathrm{d}. \tag{7}$$

played by various woodwinds. The first three chords, $\mathrm{G} \to \mathrm{E} \to \mathrm{d}$, is a transposition by T_7 of the first three chords in the opening, $\mathrm{C} \to \mathrm{A} \to \mathrm{g}$. Moreover, the sequence of progressions in (7) is a diminution of the longer sequence in (2), transposed by T_7. In fact, the last 5 chords:

$$\mathrm{G} \to \mathrm{E} \to \mathrm{d} \to \mathrm{E} \to \mathrm{d}$$

underlies a melody carried in the strings (violins, violas, and cellos), which is a slight diminution of the melody for the English horn melody in the opening.[6] Compared with the music in Example 1, the music here has been transposed up a fifth by applying T_7. Therefore, the analysis we gave in the first example can be simply transposed on the *Tonnetz*, by applying T_7. On the *Tonnetz*, the transposition T_7 is a vertical shift across one hexagon. We show the transposed diagrams in Figure 3. The vertical shifting from Figure 1 is clear.

In this third example, the factorization $\mathrm{E} \xrightarrow{\mathcal{R}} \mathrm{c}^\sharp \xrightarrow{\mathsf{T}_1} \mathrm{d}$ does have a clear musical significance. The last progression in (7), $\mathrm{E} \to \mathrm{d}$, provides the harmony for the melody in the strings (excepting the contrabasses). The chord E harmonizes the end of the transposed, diminished melody from the English horn. The chord d harmonizes the beginning of a new melody in the strings, which we shall see below is in A-Aeolian mode. The last note of this transposed, diminished melody as played by the strings is $\mathrm{C}_4^\sharp$, while the next note for the violins and violas is A_3. This jump down by a major third in those strings' melodies implies that $\mathrm{C}^\sharp$ is neither a passing tone nor a chromatic neighbor tone. Instead, it is a clear manifestation, within the surface structure, of the factorization $\mathrm{E} \xrightarrow{\mathcal{R}} \mathrm{c}^\sharp \xrightarrow{\mathsf{T}_1} \mathrm{d}$. The note $\mathrm{C}^\sharp$ is a consequence of the intermediary, underlying chord $\mathrm{c}^\sharp$, within this factorization.[7]

[6] Pike (2003, p. 186) observes that "the woodwind and strings are reversed as compared with the opening."

[7] The cellos, however, do play the $\mathrm{C}_4^\sharp$ note as an intermediate chromatic tone leading to D_4. This prevents the listener from hearing a rather jarring jump in the melody played by the other strings, a jump down from $\mathrm{C}_4^\sharp$ to A_3.

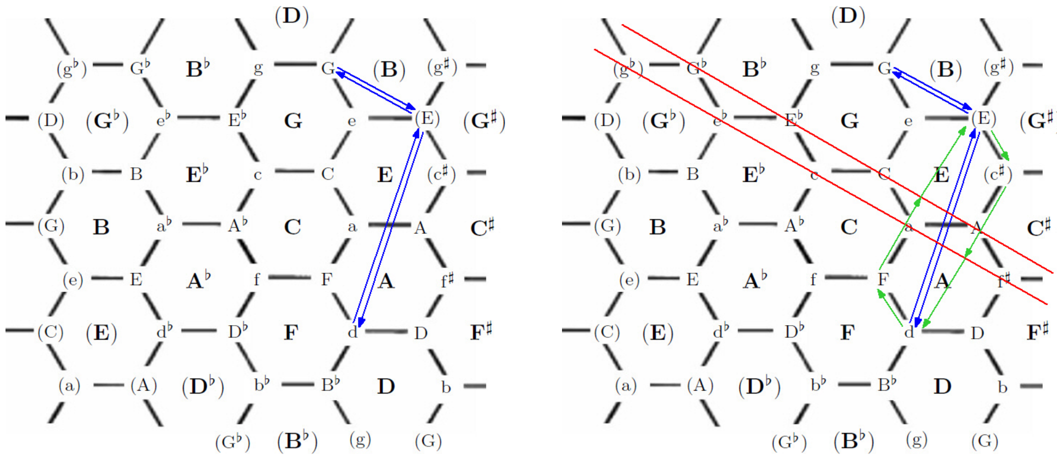

Figure 3. Left: Moves on the *Tonnetz* corresponding to transposed portion of horn melody to strings. Right: Factorization of the progressions $\mathrm{E} \to \mathrm{d}$ and $\mathrm{d} \to \mathrm{E}$, using geometric operations on the *Tonnetz*.

1.4 Modal Aspects of *Romanza*

An important aspect of *Romanza* is its use of modes for various folk-like melodies, which occur in rapid sequence throughout the piece. This modal aspect of *Romanza* is frequently cited. See, e.g., the books by Lionel Pike (2003) and Elliott Schwartz (1964). Here, we will identify some of the modes used for the melodies, and show how they relate to the Table of Diatonic Relations constructed by Ian Bates in his Ph.D. thesis (Bates, 2008) and one of his papers (Bates, 2012).

We now describe the first several modal melodies occurring in *Romanza*. The first melody belongs to the English horn. This melody is in the E-Phrygian mode for the following reasons. The melody begins with a succession of identical C-notes, then passes through a D note to two longer held E-notes that occupy the whole of the second measure. These longer, emphasized E-notes, establish the tonic of E. In fact, the melody goes up and down around this central E, and concludes with two E notes. Since the melody contains no sharped or flatted notes (except for one $\mathrm{F}^\sharp$ note, which we discussed in Example 1), the scale is C-major, but the tonic is E, which is the E-Phrygian mode.[8]

After the horn melody concludes, the music converts to a many-voiced contrapuntal texture. A solo cello plays one melody at a higher dynamic level—**mp**, as opposed to **p** or **pp** for the other instruments— beginning with measure 12. For similar reasons as we gave for the English horn melody, this solo cello melody is in the A-Aeolian mode. In fact, as pointed out in the book by Schwartz (1964, p. 93), the whole texture of the symphony at this point is A-Aeolian. The pitch classes for measures 12 through 28 trace rapidly through a sequence of chords (some of which are sustained, others passing only briefly). If we classify these chords based only on their roots, we have this sequence of 26 progressions:[9]

$$\begin{aligned}
&\mathrm{e} \xrightarrow{1} \mathrm{F} \xrightarrow{2} \mathrm{C} \xrightarrow{3} \mathrm{d} \xrightarrow{4} \mathrm{a} \xrightarrow{5} \mathrm{d} \xrightarrow{6} \mathrm{e} \xrightarrow{7} \mathrm{a} \xrightarrow{8} \mathrm{F} \xrightarrow{9} \\
&\mathrm{G} \xrightarrow{10} \mathrm{e} \xrightarrow{11} \mathrm{d} \xrightarrow{12} \mathrm{a} \xrightarrow{13} \mathrm{e} \xrightarrow{14} \mathrm{F} \xrightarrow{15} \mathrm{a} \xrightarrow{16} \mathrm{e} \xrightarrow{17} \\
&\mathrm{a} \xrightarrow{18} \mathrm{e} \xrightarrow{19} \mathrm{C} \xrightarrow{20} \mathrm{a} \xrightarrow{21} \mathrm{F} \xrightarrow{22} \mathrm{d} \xrightarrow{23} \mathrm{F} \xrightarrow{24} \mathrm{d} \xrightarrow{25} \mathrm{F} \xrightarrow{26} \mathrm{d}.
\end{aligned} \tag{8}$$

The fact that the chord a, which always appears as the A-minor triad, is the tonic chord is made quite clear by

[8]As we mentioned in a footnote to Remark 1, one could view the English horn melody as a merging of E-Phrygian and E-Aeolian modes. We shall instead emphasize the view that the presence of the single $\mathrm{F}^\sharp$ note is a surface manifestation of the factorization discussed in Example 1, as it simplifies the discussion (although the interested reader could easily modify our account in order to incorporate this mode merging).

[9]The progressions in (8) appear most clearly in the *Tonnetz* animation at the link in (1).

a diagram of these chord changes on the *Tonnetz*. (See Figure 4.)

From this diagram, we can clearly see the central role played by the chord a. Therefore, since there are also no sharps or flats on any notes in this passage, it follows that the whole symphonic texture of this passage is A-Aeolian. It is also interesting that the vast majority of the chords in the diagram are all directly related to the central chord a, via one *Tonnetz* transformation, along the chordal spine containing a. This is consistent with the analysis of modes given in Section 7.4.3 of Walker and Don (2013); see especially the last line of Table A.1 on p. 289.

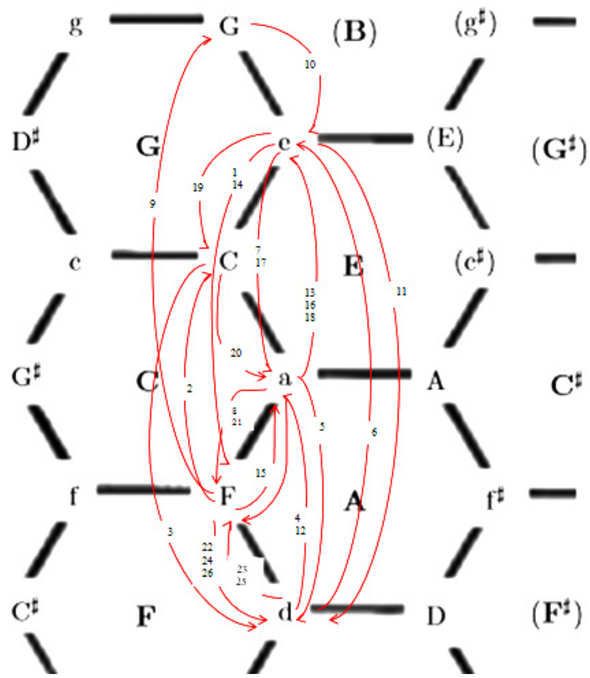

Figure 4. Diagram of the motion of the chord changes in (8).

A transition back to an E-Phrygian mode occurs with measure 32. From measures 31 through 38, there is a fugal combination of melodies played by the flutes, an oboe, clarinets in A, and English horn. These melodies are all in the E-Phrygian mode—due to emphasized starting notes of E and highest pitch notes of E.

At measure 39, a very brief modal ambiguity occurs. This is where the harmony described in Example 3 occurs. The melody is also a transposition by T_7 of a portion of the English horn melody from the opening. Since the passage alludes to the original melody, with a tonic of $\mathbf{E}$, this new passage would then have a tonic of $\mathbf{B}$. Since no sharps or flats are used, the mode would then be B-Locrian. The passage is ambiguous, however, since T_7 applied to the mode E-Phrygian produces the mode B-Phrygian. We will discuss this point further below. The only difference between the scales for B-Locrian and B-Phrygian is the F note for the Locrian mode, versus an $\mathrm{F}^\sharp$ note for the Phrygian mode. But, neither F nor $\mathrm{F}^\sharp$ appears in the melody, and that creates the ambiguity. It is worth noting, however, that two F-notes do occur immediately following the $\mathrm{C}^\sharp$ note as harmony notes. There is an F_5 note in the harmony played by the first flute section, and there is also an F_3 note played by the second viola section. This F_3 note lies below the A_3 note in the melody line for those same violas, which it is harmonizing. These F notes, along with the allusion to the English horn melody, and the almost total absence of sharps and flats throughout the piece, give some justification for hearing a hint of B-Locrian mode from the strings' melody.

We conclude our brief analysis of modes with a discussion of the change of mode at measure 47 (rehearsal mark 3). This is where the chord d occurs, in the factorized progression $\mathrm{E} \xrightarrow{\mathcal{R}} \mathrm{c}^\sharp \xrightarrow{\mathsf{T}_1} \mathrm{d}$ discussed above in

	$7\flat$	$6\flat$	$5\flat$	$4\flat$	$3\flat$	$2\flat$	$1\flat$	0	$1\sharp$	$2\sharp$	$3\sharp$	$4\sharp$	$5\sharp$	$6\sharp$	$7\sharp$
Ionian	$C^{\flat}_{I}$	$G^{\flat}_{I}$	$D^{\flat}_{I}$	$A^{\flat}_{I}$	$E^{\flat}_{I}$	$B^{\flat}_{I}$	F_{I}	C_{I}	G_{I}	D_{I}	A_{I}	E_{I}	B_{I}	$F^{\sharp}_{I}$	$C^{\sharp}_{I}$
Mixolyd.	$G^{\flat}_{M}$	$D^{\flat}_{M}$	$A^{\flat}_{M}$	$E^{\flat}_{M}$	$B^{\flat}_{M}$	F_{M}	C_{M}	G_{M}	D_{M}	A_{M}	E_{M}	B_{M}	$F^{\sharp}_{M}$	$C^{\sharp}_{M}$	$G^{\sharp}_{M}$
Dorian	$D^{\flat}_{D}$	$A^{\flat}_{D}$	$E^{\flat}_{D}$	$B^{\flat}_{D}$	F_{D}	C_{D}	G_{D}	D_{D}	A_{D}	E_{D}	B_{D}	$F^{\sharp}_{D}$	$C^{\sharp}_{D}$	$G^{\sharp}_{D}$	$D^{\sharp}_{D}$
Aeolian	$A^{\flat}_{A}$	$E^{\flat}_{A}$	$B^{\flat}_{A}$	F_{A}	C_{A}	G_{A}	D_{A}	A_{A}	E_{A}	B_{A}	$F^{\sharp}_{A}$	$C^{\sharp}_{A}$	$G^{\sharp}_{A}$	$D^{\sharp}_{A}$	$A^{\sharp}_{A}$
Phrygian	$E^{\flat}_{P}$	$B^{\flat}_{P}$	F_{A}	C_{P}	G_{P}	D_{P}	A_{P}	E_{P}	B_{P}	$F^{\sharp}_{P}$	$C^{\sharp}_{P}$	$G^{\sharp}_{P}$	$D^{\sharp}_{P}$	$A^{\sharp}_{P}$	$E^{\sharp}_{P}$
Locrian	$B^{\flat}_{Lo}$	F_{Lo}	C_{Lo}	G_{Lo}	D_{Lo}	A_{Lo}	E_{Lo}	B_{Lo}	$F^{\sharp}_{Lo}$	$C^{\sharp}_{Lo}$	$G^{\sharp}_{Lo}$	$D^{\sharp}_{Lo}$	$A^{\sharp}_{Lo}$	$E^{\sharp}_{Lo}$	$B^{\sharp}_{Lo}$
Lydian	$F^{\flat}_{Ly}$	$C^{\flat}_{Ly}$	$G^{\flat}_{Ly}$	$D^{\flat}_{Ly}$	$A^{\flat}_{Ly}$	$E^{\flat}_{Ly}$	$B^{\flat}_{Ly}$	F_{Ly}	C_{Ly}	G_{Ly}	D_{Ly}	A_{Ly}	E_{Ly}	B_{Ly}	$F^{\sharp}_{Ly}$
Ionian	$C^{\flat}_{I}$	$G^{\flat}_{I}$	$D^{\flat}_{I}$	$A^{\flat}_{I}$	$E^{\flat}_{I}$	$B^{\flat}_{I}$	F_{I}	C_{I}	G_{I}	D_{I}	A_{I}	E_{I}	B_{I}	$F^{\sharp}_{I}$	$C^{\sharp}_{I}$

Figure 5. Table of Diatonic Relations. The top row of modes is a repetition of the bottom row of modes. The $7\sharp$ column is enharmonically equivalent with the $5\flat$ column. Moreover, the $7\flat$ column is enharmonically equivalent with the $5\sharp$ column, and the $6\flat$ column is enharmonically equivalent with the $6\sharp$ column. The rectangular array of modes, lying between the horizontal and vertical lines, marks off all the enharmonically distinct diatonic modes. (The modes in this table can be viewed as points that are uniformly spaced on a flat torus, similar to how the *Tonnetz* can be viewed.)

Example 3. The chord d is clearly emphasized here, and the highest pitch note in the separate, contrapuntally combined melodies, is also a D note. The mode of D-Dorian is thereby established. In fact, the passage that begins here is quite similar, but in a different mode, to the A-Aeolian contrapuntal passage discussed above.

We have seen that, during the course of the first half or so of the movement, Vaughan Williams has used a sequence of modes with different tonic notes. The sequence of clearly established modes is E-Phrygian, A-Aeolian, E-Phrygian, D-Dorian. No clear, overall tonic is ever established.[10] We shall now show how all of these modes are related through the *Table of Diatonic Relations.*

Table of Diatonic Relations

There is a systematic relationship between the various modes that we have analyzed in *Romanza*. This relationship is specified by how the modes appear in the Table of Diatonic Relations, first constructed by Ian Bates in his thesis (Bates, 2008). In Figure 5 we show the Table of Diatonic Relations.[11]

In this table, a mode is indicated by the note for its tonic, with a subscript identifying its mode type. For example, C_I denotes the C-Ionian mode. The scale for C_I is the C-major scale:

$$C_I\!: \quad C \ D \ E \ F \ G \ A \ B \ C.$$

As another example, $E^{\flat}_{M}$ denotes the $E^{\flat}$-Mixolydian mode. The scale for $E^{\flat}_{M}$ is

$$E^{\flat}_{M}\!: \quad E^{\flat} \ F \ G \ A^{\flat} \ B^{\flat} \ C \ D^{\flat} \ E^{\flat}.$$

The table shows a way of viewing how close different modes are to each other. Modes that are adjacent horizontally are related by either transposing up or down a fifth, while modes that are adjacent vertically are related by either diatonically shifting up or down a fifth. For example, the modes A_A and E_P are in the same column as C_I. The scale for E_P is

$$E_P\!: \quad E \ F \ G \ A \ B \ C \ D \ E.$$

[10]It is interesting, however, that the tonic of D in the D-Dorian mode is a rather delayed version of the missing tonic from the opening D-minor chords. The chord d was also emphasized at the start of this passage, and repeatedly at the end of the chords in (8).

[11]The table shown in Figure 5 is equivalent to the one Bates constructed; we have just adapted ours to better display the relationships of the modes in *Romanza*.

By applying the diatonic scale shift $\mathcal{S}_{-4}$ to this scale—which means going down a perfect fifth—we obtain the scale for A_A:

$$\mathrm{A}_A\text{:}\quad \mathrm{A\ B\ C\ D\ E\ F\ G\ A.}$$

It is worth noting that this application of $\mathcal{S}_{-4}$ is just a rotation by 4 hours counter-clockwise on the 7-hour diatonic clock given in Figure 3.18 on p. 70 of Walker and Don (2013). In fact, there is nothing special about the C-major scale, moving up one row vertically from any given mode corresponds to applying $\mathcal{S}_{-4}$ to the scale for that mode. Since going up vertically corresponds to applying $\mathcal{S}_{-4}$, it follows that going down vertically corresponds to applying $\mathcal{S}_4$. Applying $\mathcal{S}_4$ corresponds to moving up a perfect fifth. We leave it as an exercise for the reader to check that moving over one column to the right from a given mode corresponds to applying the transposition T_7, which is transposing up a fifth, and that moving one column to the left from a given mode corresponds to applying the transposition T_{-7} which is transposing down a fifth.

The remarks in the previous paragraph show how the table is constructed. Begin at the bottom of the table with the C_I mode in column 0 and the Ionian row. Apply $\mathcal{S}_{-4}$ repeatedly until, after 7 applications, the mode returns to C_I in the top row. (The mode returns to C_I because we are rotating around a 7-hour clock, i.e. using mod 7 arithmetic.) Applying T_7 to each mode in column 0, we get the modes in the $1\sharp$ column. Likewise, applying T_{-7} to the modes in column 0, gives the modes in the $1\flat$ column. Repeatedly applying T_7 produces the modes for all the columns to the right of the 0 column, and repeatedly applying T_{-7} produces the modes for all the columns to the left of the 0 column, just like in the circle of fifths. Moreover, just like in the circle of fifths, the columns at the right eventually match up with the columns on the left (if we invoke enharmonic equivalence). For example, the scale for $\mathrm{D}^\sharp_D$ in column $7\sharp$ is enharmonically equivalent with the scale for $\mathrm{E}^\flat_D$ in column $5\flat$, since they are both Dorian modes, and their tonics of $\mathrm{D}^\sharp$ and $\mathrm{E}^\flat$ are enharmonic. The connection with the circle of fifths shows why the table organizes modes according to their musical relatedness. Traditionally, keys that are close to each other on the circle of fifths are thought to be near to each other in a musical sense. A smooth modulation between keys would be to change between two keys that are near to each other on the circle of fifths. Similarly, a smooth shift of modes would result when the two modes are near to each other in the horizontal direction. Two modes that are near to each other in a vertical direction also involve a change by a fifth, so the theory is that they would also be close musically. A smooth shift of modes would occur when the two modes are near to each other in the vertical direction. Consequently, the theory is that any two modes that are near to each other in the table are close to each other musically, so a shift of modes between them should sound musically smooth. The proof of this theory is how well it applies to musical examples. We now show how it applies to *Romanza*, and mention some of its other applications.

In the discussion of modes in *Romanza*, we found the following shifting of modes:

$$\text{E-Phrygian} \to \text{A-Aeolian} \to \text{E-Phrygian} \to \text{D-Dorian} \tag{9}$$

and a somewhat ambiguous hinting of B-Locrian also occurred. If we include this mode change, we have the following shiftings of modes:

$$\mathrm{E}_P \to \mathrm{A}_A \to \mathrm{E}_P \to \mathrm{B}_{Lo} \to \mathrm{D}_D. \tag{10}$$

These four mode changes all occur along four adjacent rows in column 0 of the Table of Diatonic Relations. See Figure 6. Thus, we have a theoretical explanation for how the music smoothly shifts between these modes.

Other examples of the use of the Table of Diatonic Relations can be found in Bates' Ph.D. thesis and his paper, both cited earlier. In those references, Bates shows how there are other closely related modes within Vaughan Williams' music. Typically, those modes lie along diagonal lines within the Table of Diatonic Relations. For example, Bates (2012, Example 13) analyzes the following changes of modes in Vaughan Williams' *Five Variants of "Dives and Lazarus"*:

$$\mathrm{B}_A \to \mathrm{B}_D \to \mathrm{B}_M \tag{11}$$

	$7\flat$	$6\flat$	$5\flat$	$4\flat$	$3\flat$	$2\flat$	$1\flat$	0	$1\sharp$	$2\sharp$	$3\sharp$	$4\sharp$	$5\sharp$	$6\sharp$	$7\sharp$
Ionian	$\mathrm{C}^\flat_I$	$\mathrm{G}^\flat_I$	$\mathrm{D}^\flat_I$	$\mathrm{A}^\flat_I$	$\mathrm{E}^\flat_I$	$\mathrm{B}^\flat_I$	F_I	C_I	G_I	D_I	A_I	E_I	B_I	$\mathrm{F}^\sharp_I$	$\mathrm{C}^\sharp_I$
Mixolyd.	$\mathrm{G}^\flat_M$	$\mathrm{D}^\flat_M$	$\mathrm{A}^\flat_M$	$\mathrm{E}^\flat_M$	$\mathrm{B}^\flat_M$	F_M	C_M	G_M	D_M	A_M	E_M	B_M	$\mathrm{F}^\sharp_M$	$\mathrm{C}^\sharp_M$	$\mathrm{G}^\sharp_M$
Dorian	$\mathrm{D}^\flat_D$	$\mathrm{A}^\flat_D$	$\mathrm{E}^\flat_D$	$\mathrm{B}^\flat_D$	F_D	C_D	G_D	D_D	A_D	E_D	B_D	$\mathrm{F}^\sharp_D$	$\mathrm{C}^\sharp_D$	$\mathrm{G}^\sharp_D$	$\mathrm{D}^\sharp_D$
Aeolian	$\mathrm{A}^\flat_A$	$\mathrm{E}^\flat_A$	$\mathrm{B}^\flat_A$	F_A	C_A	G_A	D_A	A_A	E_A	B_A	$\mathrm{F}^\sharp_A$	$\mathrm{C}^\sharp_A$	$\mathrm{G}^\sharp_A$	$\mathrm{D}^\sharp_A$	$\mathrm{A}^\sharp_A$
Phrygian	$\mathrm{E}^\flat_P$	$\mathrm{B}^\flat_P$	F_A	C_P	G_P	D_P	A_P	E_P	B_P	$\mathrm{F}^\sharp_P$	$\mathrm{C}^\sharp_P$	$\mathrm{G}^\sharp_P$	$\mathrm{D}^\sharp_P$	$\mathrm{A}^\sharp_P$	$\mathrm{E}^\sharp_P$
Locrian	$\mathrm{B}^\flat_{Lo}$	F_{Lo}	C_{Lo}	G_{Lo}	D_{Lo}	A_{Lo}	E_{Lo}	B_{Lo}	$\mathrm{F}^\sharp_{Lo}$	$\mathrm{C}^\sharp_{Lo}$	$\mathrm{G}^\sharp_{Lo}$	$\mathrm{D}^\sharp_{Lo}$	$\mathrm{A}^\sharp_{Lo}$	$\mathrm{E}^\sharp_{Lo}$	$\mathrm{B}^\sharp_{Lo}$
Lydian	$\mathrm{F}^\flat_{Ly}$	$\mathrm{C}^\flat_{Ly}$	$\mathrm{G}^\flat_{Ly}$	$\mathrm{D}^\flat_{Ly}$	$\mathrm{A}^\flat_{Ly}$	$\mathrm{E}^\flat_{Ly}$	$\mathrm{B}^\flat_{Ly}$	F_{Ly}	C_{Ly}	G_{Ly}	D_{Ly}	A_{Ly}	E_{Ly}	B_{Ly}	$\mathrm{F}^\sharp_{Ly}$
Ionian	$\mathrm{C}^\flat_I$	$\mathrm{G}^\flat_I$	$\mathrm{D}^\flat_I$	$\mathrm{A}^\flat_I$	$\mathrm{E}^\flat_I$	$\mathrm{B}^\flat_I$	F_I	C_I	G_I	D_I	A_I	E_I	B_I	$\mathrm{F}^\sharp_I$	$\mathrm{C}^\sharp_I$

Figure 6. Changes of Modes in Bates' Table of Diatonic Relations. Within the rectangle, are the changes of modes in Vaughan Williams' *Romanza* shown in (10). The diagonal mode changes $\mathrm{B}_A \rightarrow \mathrm{B}_D \rightarrow \mathrm{B}_M$ are described in (11). The diagonal mode change $\mathrm{C}^\sharp_A \rightarrow \mathrm{C}^\sharp_D$ is described in (12).

This is a diagonal movement between adjacent modes, as shown in Figure 6.

Another instance of diagonal movement occurs in the opening of Debussy's *Sarabande* movement from his *Pour le Piano*. Anton East (2012) describes the modes that occur in the beginning of *Sarabande* as follows:

> The music sounds neither major nor minor, yet the important cadences at the end of the first page[12] and the end of the piece confirm C sharp as the tonic. With this in mind the opening two lines would suggest the key is the aeolian mode on C sharp as there are no raised 7th notes (B sharps) that would make the music minor. The A sharps in the third line move the music into the dorian mode, a mode close to the aeolian but with a brighter sixth degree of the scale.

In other words, East has analyzed the following change of modes in the opening of *Sarabande:*

$$\mathrm{C}^\sharp_A \rightarrow \mathrm{C}^\sharp_D. \tag{12}$$

These modes are related diagonally on the Table of Diatonic Relations, as shown in Figure 6. These adjacent diagonal mode changes are all examples of $1\sharp$ ***signature transformations*** as described by Julian Hook (2008). Our research group has also been exploring how Bates' Table of Diatonic Relations applies to modal aspects of the music of Prokofiev. Our preliminary results show a number of uses of signature transformations.

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[12] *Sarabande*'s score can be obtained from here.

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