# Denoising Gabor Transforms

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***Abstract*— We describe denoising one-dimensional signals by thresholding Blackman windowed Gabor transforms. This method is compared with Gauss-windowed Gabor threshold denoising and wavelet-based denoising, and is found to be superior in most cases. A new, localized estimator of noise standard deviation is also obtained. Our work provides the first step in developing an adaptive denoising method for non-stationary noise.**

***Index Terms*— Signal denoising, Gabor transforms, thresholding. EDICS: DSP-RECO**

## INTRODUCTION

The Gabor transform is a classic method of time-frequency analysis, [2], [8], [9], [12]. Recent work has focused on its advantages for discrete signal processing [3], [5], [13]–[16]. The work in [5], in particular, applies discrete Gabor transforms to denoising. The window function used in [5], however, is the non-compactly supported Gaussian function (the one introduced by Gabor). The non-compact support of the window requires elaborate techniques, such as periodization and the Zak transform, to be brought to bear in the discrete realm. We study in this paper the compactly supported Blackman window for discrete Gabor transforms; comparing its denoising performance with the method proposed in [5] and with the wavelet-based methods of SureShrink [7] and BayesShrink [4]. We find that it outperforms the method in [5] for almost all tested signals and outperforms the wavelet methods on most tested signals (see Table III).

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The method proposed here, thresholding Blackman windowed Gabor transforms, has precursors. Gabor transform thresholding for denoising was done in [11] and [17]. However, they used iterations. In [11], 100 iterations were used, while in [17], 10 iterations were used along with prior information on the transmitted signal (that it was a chirp); and neither used compactly supported Blackman windows. There may be applications where speed is critical and the one-step, compactly supported, approach described here would be advantageous. We also have developed a new, local estimator of the noise standard deviation.

In section I we define the discrete Gabor transform with a Blackman window and derive our denoising method. We also state a theorem on convergence of the proposed method and provide some justification for using a Blackman window. Section II contains an objective comparison, using Mean Square Error (MSE), of the performance of the proposed denoising method with the other non-iterative denoising methods mentioned above. In Section III we prove the convergence theorem stated in section I. We conclude with a summary and directions for future research.

## I. DERIVATION OF THE METHOD

We assume that the signal $f(t_k)$, for values $t_k$ uniformly spaced by a constant increment $\Delta t$, satisfies the following additive noise model:

$$f(t_k) = g(t_k) + \mathrm{n}_k \tag{1}$$

where $g$ is the underlying signal that is corrupted by the additive noise $\{\mathrm{n}_k\}$. To keep things simple in this initial proposal, we assume that the underlying signal $g$ is *piecewise regular.* By piecewise regular, we mean that $g$ and $g'$ are bounded with only finitely many jump discontinuities, and $g''$ is integrable with only finitely many discontinuities. We shall also assume that the values $\{\mathrm{n}_k\}$ are independently identically distributed Gaussian random variables of mean $0$ and standard deviation $\sigma$. [We shall write $\mathrm{n}_k \sim \mathcal{N}(0, \sigma^2)$.]

A Gabor transform of $f$, with window function $w$, is defined as follows. First, multiply $\{f(t_k)\}$ by a sequence of shifted window functions $\{w(t_k - \tau_m)\}$, producing a sequence of time localized subsignals, $\{f(t_k)w(t_k - \tau_m)\}_{m=1}^{M}$. Uniformly spaced time values $\{\tau_m\}_{m=1}^{M}$ are used for the shifts. The windows $\{w(t_k - \tau_m)\}$ are all compactly supported and overlap each other. Second, because $w$ is compactly supported we treat each subsignal $\{f(t_k)w(t_k - \tau_m)\}$ as a finite sequence and apply an $N$-point FFT $\mathcal{F}$ to it. This yields the Gabor transform of $\{f(t_k)\}$:

$$\{\mathcal{F}\{f(t_k)w(t_k - \tau_m)\}\}_{m=1}^{M}. \tag{2}$$

It is important to note that when the values $t_k$ belong to a finite interval, we shall always evenly extend signal values from each endpoint, so that the full supports of all the windows are included.

When displaying a Gabor transform, it is standard practice to display a *spectrogram*, a plot of its magnitude-squared values, with time along the horizontal axis, frequency along the vertical axis, and darker pixels representing higher square-magnitudes. For example, see Fig. 2.

The Gabor transform for our proposed denoising method uses a *Blackman window* defined by $w(t) = 0.42 + 0.5\cos(\frac{2\pi t}{\lambda}) + 0.08\cos(\frac{4\pi t}{\lambda})$ for $|t| \leq \lambda/2$, and $w(t) = 0$ for $|t| > \lambda/2$, for a positive parameter $\lambda = N\Delta t$ equalling the width of the window where the FFT is performed. Another popular window is a *Hanning window.* We focus on the Blackman window $w$ since it generally gives slightly lower MSEs (although the differences were so insignificant that we have chosen not to report them), and because the spectrograms of some denoised signals show more leakage with Hanning windowing than with Blackman windowing. The Fourier transform of the Blackman window is very nearly positive (with negative values less than $10^{-4}$ in size), thus providing an effective substitute for a Gaussian function (which is well-known to have minimum time-frequency support). Further evidence of the superiority of Blackman-windowing is summarized in Table II of [1].

Our denoising method uses the inversion process for Gabor transforms, which we now review. For the windowings we employ, the windows are greatly overlapping. In fact, there is a constant $A > 0$ ($A > 1/2$ for our method) such that

$$A \leq \sum_{m=1}^{M} w^2(t_k - \tau_m) \tag{3}$$

holds for all points $t_k$. We use (3) to invert the Gabor transform in (2). First, apply inverse FFTs to all the transforms $\{\mathcal{F}\{f(t_k)w(t_k - \tau_m)\}\}_{m=1}^{M}$ to obtain $\{\{f(t_k)w(t_k - \tau_m)\}\}_{m=1}^{M}$. Then multiply each subsignal $\{f(t_k)w(t_k - \tau_m)\}$ by $\{w(t_k - \tau_m)\}$ and sum over $m$, obtaining $\{f(t_k)\sum_{m=1}^{M} w^2(t_k - \tau_m)\}$. Multiplying by $[\sum_{m=1}^{M} w^2(t_k - \tau_m)]^{-1}$, which is no larger than $A^{-1}$, we obtain $\{f(t_k)\}$. Thus, we can stably invert our Gabor transforms.

We now recall the effect of applying an FFT on the i.i.d. Gaussian noise $\{\mathrm{n}_k\}$. The FFT $\mathcal{F}$ that we employ is given by, for all sequences $\{a_k\}_{k=0}^{N-1}$:

$$\{a_k\}_{k=0}^{N-1} \xrightarrow{\mathcal{F}} \{A_\ell = \sum_{k=0}^{N-1} a_k e^{-i2\pi k\ell/N}\}.$$

From *Parseval's equality*

$$\sum_{k=0}^{N-1} |a_k|^2 = \frac{1}{N} \sum_{\ell=0}^{N-1} |A_\ell|^2 \tag{4}$$

we conclude that $\mathcal{F}$ is $\sqrt{N}$ times a unitary transformation. It is well-known that for Gaussian i.i.d noise, $\{\mathrm{n}_k\}$, the real part of the FFT, $\{\Re(\mathrm{N}_\ell)\}$, and its imaginary part, $\{\Im(\mathrm{N}_\ell)\}$, satisfy (on average, in terms of expected value):

$$\sum_{\ell=0}^{N-1} |\Re(\mathrm{N}_\ell)|^2 = \sum_{\ell=0}^{N-1} |\Im(\mathrm{N}_\ell)|^2 = \frac{1}{2} \sum_{\ell=0}^{N-1} |\mathrm{N}_\ell|^2. \tag{5}$$

From equations (4) and (5), we conclude that both $\Re\mathcal{F}$ and $\Im\mathcal{F}$ can be expected to behave on our noise as $\sqrt{N/2}$ times orthogonal transformations. Hence $\{\Re\mathcal{F}(\mathrm{n}_k)\}$ and $\{\Im\mathcal{F}(\mathrm{n}_k)\}$ will be i.i.d. normal with mean $0$ and standard deviation $\sigma\sqrt{N/2}$.

We now examine the effect of multiplying the noise signal $\{\mathrm{n}_k\}$ by a Blackman window $\{w(t_k - \tau_m)\}$. For simplicity of notation, and because the noise is stationary, we set $\tau_m = 0$ and $w_k = w(t_k)$. By repeatedly computing the mean and standard deviation of $\{\mathrm{n}_k w_k\}$, we conclude that the mean of $\{\mathrm{n}_k w_k\}$ is still near $0$ (always less than 1/100 the standard deviation) and the standard deviation is approximately $0.55\sigma$. Theoretically, we explain this result as follows. The expected value of the variance estimation

$$\frac{1}{N-1} \sum_{k=0}^{N-1} w(t_k)^2 \mathrm{n}_k^2 \approx \sum_{k=0}^{N-1} w(t_k)^2 \frac{\mathrm{n}_k^2}{N}$$

is $\sum_{k=0}^{N-1} w(t_k)^2\sigma^2/N$, a Riemann sum approximation of $\int_{-1/2}^{1/2} w(t)^2\sigma^2\, dt = (0.55)^2\sigma^2$. Similar calculations show that the expected value of the mean is approximately $0$ as well (we made use of this fact in the preceding sentence). Thus, we expect that the standard deviation of $\{\mathrm{n}_k w_k\}$ is approximately $0.55\sigma$, and its mean is approximately $0$, as confirmed by our repeated numerical experiments.

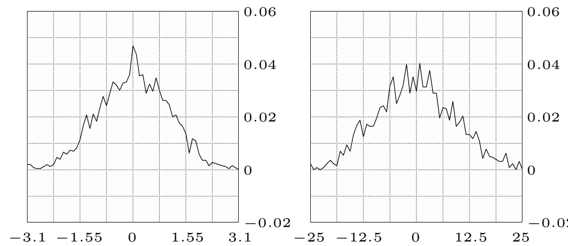


Fig. 1. Left: Average of 5 histograms of Gaussian random noise, 512 samples with variance 1. Right: Average of 5 histograms of real part of FFT of Blackman windowing of the random noise data, calculated variances were all approximately 78, and 78 $\approx (0.55)^2\,512/2$.

Since $\mathcal{F}$ converts multiplication into convolution and $w$ is an even function, we conclude that $\Re\mathcal{F}(\mathrm{n}_k w_k)$ and $\Im\mathcal{F}(\mathrm{n}_k w_k)$ are convolutions of $\Re\mathcal{F}(\mathrm{n}_k)$ and $\Im\mathcal{F}(\mathrm{n}_k)$ with $\mathcal{F}(w_k)$. Since $\mathcal{F}(w_k)$ is of very narrow width—having non-zero values of $0.42N$, $0.25N$, and $0.04N$ only at the frequency indices $\ell = 0$, $\pm1$, and $\pm2$, respectively—with a mean over $N$ indices of $1$, we shall assume that these convolutions are still normally distributed. The means of $\Re\mathcal{F}(\mathrm{n}_k w_k)$ and $\Im\mathcal{F}(\mathrm{n}_k w_k)$ will be approximately $0$ and their variances will be approximately $(0.55)^2\sigma^2\,N/2$. In Fig. 1, we show histograms illustrating this result. The averaged histogram of $\{\mathrm{n}_k\}$ approximates a normal distribution, and so does the averaged histogram of $\Re\mathcal{F}(\mathrm{n}_k w_k)$. Furthermore, we find that the calculated variances for $\Re\mathcal{F}(\mathrm{n}_k w_k)$ were all about $78 \approx (0.55)^2\,N/2$ when $N = 512$. Similar results obtain for $\Im\mathcal{F}(\mathrm{n}_k w_k)$.

We can now derive our method. First, we note that the normalized data $\Re\mathcal{F}(\mathrm{n}_k w_k)/(0.55\sigma\sqrt{N/2})$ is $\mathcal{N}(0,1)$. Hence, with probability $\to 1$ as $N \to \infty$, all of their values will be smaller in magnitude than

$\sqrt{\log N}$. (In fact, because of the rapid decrease to $0$ of $e^{-x^2/2}$ it is almost certain that all $N$ magnitudes lie below $\sqrt{\log N}$.) Similarly, we expect that all (certainly all as $N \to \infty$) of the magnitudes of $\Im\mathcal{F}(\mathrm{n}_k w_k)/(0.55\sigma\sqrt{N/2})$ lie below $\sqrt{\log N}$. We conclude that *for a threshold of* $T = 0.55\sigma\sqrt{N \log N}$ *we expect that all Gabor transformed noise values will have magnitudes no larger than* $T$. Based on this conclusion, our denoising method is the following four-step process:

1) Compute a Blackman-windowed Gabor transform of $\{f(t_k)\}$ using $N = c\sqrt{N_s}$ length FFTs (see Table II);
2) Obtain an estimate $\widehat{\sigma}$ of the standard deviation $\sigma$ of the noise (see Remark 1 below);
3) Replace all Gabor transform values of magnitude less than the threshold $T = 0.55\widehat{\sigma}\sqrt{N \log N}$ by zero-values (*hard thresholding*);
4) Apply the inversion procedure to the thresholded Gabor transform, producing the denoised signal.

*Remark 1:* We find $\widehat{\sigma}$ by the following process. First, for each of the first $20$ windowings, we compute a median of the imaginary parts of the *highest quarter frequency* values for our Gabor transform. Second, we compute the arithmetic mean of those $20$ medians. Finally, we divide this average median by $0.6745 * 0.55 * (N/2)$. Using a median estimator on high-frequency data of orthogonal transforms has been shown to be a good method by other workers; see e.g., [6] and [4] for wavelet transforms. We used an average of $20$ windowings for stabilizing the predicted results based on simulations with pure i.i.d. Gaussian noise (some stabilization was needed due to the localizing of the median estimator). In Table I we summarize the ranges, means, and dispersion (half the difference between 1st and 3rd quartiles) for $\widehat{\sigma}$, using $60$ trials for each $N_s$ ($10$ trials each for the $6$ test functions described in section II) with additive i.i.d. Gaussian noise having $\sigma = 1$. This data shows that our estimator converges quickly with increasing sampling rate to a good approximation of $\sigma$. [The reason why is briefly explained in the first paragraph of section III.] We also note that this estimator is *local*, which sets the stage for further development of a denoising procedure for non-stationary noise.

| Samples $N_s$ | Range | Mean | Dispersion |
|---|---|---|---|
| 512 | 0.810 to 2.150 | 1.423 | 0.238 |
| 2048 | 0.819 to 1.657 | 1.129 | 0.098 |
| 8192 | 0.840 to 1.322 | 1.021 | 0.041 |

TABLE I

PROPOSED METHOD'S ST. DEV. ESTIMATES FOR $\sigma = 1$.

The proposed method does converge for a wide variety of signals as evidenced by the following theorem.

*Theorem 1: Let* $f(t_k) = g(t_k) + \mathrm{n}_k$ *where* $g$ *is piecewise regular on* $[0, 1]$ *and* $\{\mathrm{n}_k\}$ *are i.i.d. normal random variables with mean* $0$ *and st. dev.* $\sigma$. *The expected value of the MSE of a Gabor-Blackman thresholded denoising* $\widetilde{f}$ *converges to* $0$ *at rate* $\mathcal{O}(1/\sqrt{N_s})$ *as the number of samples* $N_s \to \infty$.

We will prove this theorem in Section III.

For a signal containing $N_s$ sample values, the proposed method is of $\mathcal{O}(N_s \log_2 N_s)$ complexity. This follows from the fact that each FFT has $\mathcal{O}(N_s^{1/2} \log_2 N_s)$ complexity. Hence our method is a fast procedure. It is also practical in applications like streaming audio, since the compact support of the Blackman windows requires only that a buffer of length $\mathcal{O}(\sqrt{N_s})$ be available for computing and thresholding the spectrogram as the data streams in.

## II. COMPARISON OF DENOISING METHODS

In this section, we compare our proposed method, called *Hard Gabor (Blackman)*, with three other methods: the hard thresholding method of [5], which we shall call *Hard Gabor (Gauss)*, and two wavelet based methods, *SureShrink* [7] and *BayesShrink* [4]. We

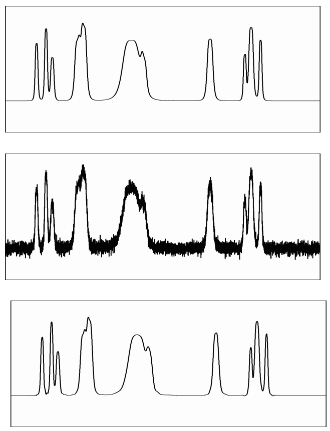

Fig. 2. Top: Bumps function, 8192 points. Middle: Bumps function with additive noise. Bottom: Denoising using Hard Gabor (Blackman) method.

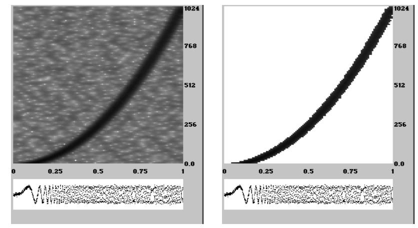

Fig. 3. Left: Spectrogram of noisy QuadChirp, 2048 points. Right: Spectrogram of its thresholded Gabor transform.

| $N_s$ | $N$ | $\Delta\tau/\Delta t$ | $M$ |
|---|---|---|---|
| 512 | 128 | 9 | 57 |
| 1024 | 128 | 9 | 114 |
| 2048 | 256 | 17 | 121 |
| 4096 | 256 | 17 | 241 |
| 8192 | 512 | 33 | 249 |
| 16384 | 512 | 33 | 497 |
| $N_s$ | $c\sqrt{N_s}$ <br> $c = 4$ or $4\sqrt{2}$ | $c\sqrt{N_s}/16 + 1$ | $\lessapprox \gamma\sqrt{N_s}$ <br> $\gamma = 4$ or $2\sqrt{2}$ |

TABLE II

PARAMETERS FOR HARD GABOR (BLACKMAN) DENOISING. $N_s$ IS THE NUMBER OF SAMPLES, $N$ IS THE FFT-LENGTH, $\Delta\tau/\Delta t$ IS THE INDEX-SHIFT, AND $M$ IS THE NUMBER OF WINDOWINGS.

compare MSEs for six test signals using data for Hard Gabor (Blackman) and SureShrink reported in [5], and data for BayesShrink generated from the MATLAB® code kindly provided for us by Eric Kolaczyk.

The six test signals are called *Bumps, HeaviSine, Doppler, Blocks, QuadChirp,* and *MishMash* in [7]. For the three sample sizes, $N_s = 512$, $N_s = 2048$, and $N_s = 8192$, these signals were obtained from uniform samples over the interval $0 \leq t < 1$. For the first four signals, Theorem 1 applies. We discuss how to extend Theorem 1 to cover QuadChirp and MishMash in Remark 2 below. In each case, the signals were normalized by multiplying by constants so that the standard deviation of the samples $\{g(t_k)\}$ was always 7. For all of the denoisings, the noisy data $\{f(t_k)\}$ was generated by adding i.i.d. Gaussian normal noise $\{\mathrm{n}_k\}$ as in (1). An example is shown in Fig. 2. Our method produces a very satisfying, almost noise-free reconstruction of the true signal (with MSE $0.015$, a 67-fold noise reduction). A second example is shown in Fig. 3. Again, the proposed method provides an excellent denoising (with MSE $0.102$, a 10-fold noise reduction).

In Table II we show the values of the parameters for the proposed method, where $N_s$ is the number of sample points, $N$ is the number of points used in the FFT $\mathcal{F}$, $\Delta\tau/\Delta t$ is the magnitude of each index shift between successive starting points for the windows $w(t_k - \tau_m)$, and $M$ is the number of windowings. To maximize speed, the parameter $c$ was chosen to yield power-of-two FFTs. Note that the parameters in Table II yield highly oversampled transforms, by a factor of $N/(\Delta\tau/\Delta t) \approx 15$. The Hard Gabor (Gauss) method in [5] used only 2-times oversampling. On the other hand, it used a pre-set exact value of $\sigma$ rather

| SIGNAL | $N_s$ | GAB-G | SSHR | BSHR | GAB-B |
|---|---|---|---|---|---|
| Bumps | 512 | **0.31** | 0.38 | 0.43 | 0.33 |
| | 2048 | 0.11 | 0.13 | 0.14 | **0.10** |
| | 8192 | 0.04 | 0.04 | 0.04 | **0.02** |
| HeaviSine | 512 | 0.25 | **0.14** | **0.14** | 0.26 |
| | 2048 | 0.10 | 0.07 | **0.05** | 0.10 |
| | 8192 | 0.05 | 0.04 | **0.02** | 0.04 |
| Doppler | 512 | 0.30 | 0.29 | **0.25** | 0.29 |
| | 2048 | 0.10 | 0.11 | 0.09 | **0.06** |
| | 8192 | 0.03 | 0.05 | 0.02 | **0.01** |
| Blocks | 512 | 0.87 | **0.49** | 0.57 | 1.13 |
| | 2048 | 0.57 | **0.25** | **0.25** | 0.58 |
| | 8192 | 0.30 | 0.11 | **0.10** | 0.28 |
| QuadChirp | 512 | 0.36 | 0.75 | 0.65 | **0.22** |
| | 2048 | 0.16 | 0.73 | 0.60 | **0.10** |
| | 8192 | 0.13 | 0.79 | 0.58 | **0.05** |
| MishMash | 512 | 0.83 | 1.11 | 1.09 | **0.41** |
| | 2048 | 0.44 | 0.56 | 1.10 | **0.23** |
| | 8192 | 0.32 | 0.42 | 1.08 | **0.13** |

TABLE III

AVERAGE MSES ON DENOISING SIX TEST SIGNALS FOR THREE NUMBERS OF SAMPLES $N_s$. GAB-G IS HARD GABOR (GAUSS), SSHR IS SURESHRINK, BSHR IS BAYESSHRINK, AND GAB-B IS HARD GABOR (BLACKMAN).

than an estimate from the noisy data.

In Table III, we report the results of our comparison of MSEs for the four methods. Averages were taken using 100 realizations of noisy signals. These results indicate the value of the proposed method, it outperforms both Hard Gabor (Gauss) and SureShrink on two-thirds of the data, and outperforms BayesShrink on slightly more than half the data. It should be noted that our proposed method is extremely simple—it does not employ advanced Bayesian statistical modelling like BayesShrink—hence there is room for improvement. It is also worth noting that the proposed method clearly outperforms the wavelet methods on two of the signals of audio type, QuadChirp and MishMash. On these two signals, the wavelet methods do not appear to be converging, while the proposed method does appear to be converging at rate $\mathcal{O}(1/\sqrt{N_s})$, as a modification of the proof of Theorem 1 does confirm.

## III. PROOF OF THEOREM 1

For simplicity of notation, we assume that the constant $c$ (see Table II) is 1. Hence $N = N_s^{1/2}$. Since Equation (7) below shows that the upper quarter of the Gabor transform coefficients tend to $0$ at an inverse square rate, it follows that our estimator $\widehat{\sigma}$ converges rapidly to $\sigma$ as $N_s \to \infty$, as indicated by the data summarized in Table I. Therefore, we shall use the exact value $\sigma$ in the calculations that follow.

For values of $t_k$ separated from any discontinuities of $g$, $g'$ and $g''$, as $N_s \to \infty$ the inversion process is adding bounded multiples of the centered value $t_k = \tau_m$. The signal $g$ has Gabor transform values $\mathcal{F}\{g(t_k)w(t_k - \tau_m)\}$ satisfying (after changes of variable to center at $\tau_m = 0$ and rescale):

$$\mathcal{F}\{g(t_k)w(t_k - \tau_m)\}[\ell] \sim \sqrt{N_s}\mathcal{F}_{\rm s}\{g(\frac{t}{\sqrt{N_s}})w(t)\}[\ell].$$

Here $\mathcal{F}_{\rm s}$ denotes Fourier series coefficients over the interval $[-1/2, 1/2]$ and the window $w$ on the right is using $\lambda = 1$. This formula was obtained by viewing the FFT as a trapezoidal rule approximation of a Fourier series integral [with a tolerable error of $\mathcal{O}(1)$ *after thresholding*].

We use

$$g(\frac{t}{\sqrt{N_s}}) \doteq g(\frac{\tau_m}{\sqrt{N_s}}) + \frac{1}{\sqrt{N_s}}g'(\frac{\tau_m}{\sqrt{N_s}})(t - \tau_m)$$

to obtain for $|\ell| \leq 2$

$$\mathcal{F}\{g(t_k)w(t_k - \tau_m)\}[\ell] \sim g(\tau_m)\mathcal{F}_{\rm s}(w)[\ell]\mathcal{O}(N_s^{1/2}) \tag{6}$$

and for $|\ell| > 2$

$$\mathcal{F}\{g(t_k)w(t_k - \tau_m)\}[\ell] \sim \mathcal{O}(\ell^{-2}). \tag{7}$$

On the other hand, the random variables $\mathcal{F}\{\mathrm{n}_k w_k\}$ satisfy

$$\begin{aligned}\Re\mathcal{F}\{\mathrm{n}_k w_k\}[\ell] &\sim \mathcal{N}(0, 0.55^2\sigma^2 N_s^{1/2}/2)\\ \Im\mathcal{F}\{\mathrm{n}_k w_k\}[\ell] &\sim \mathcal{N}(0, 0.55^2\sigma^2 N_s^{1/2}/2).\end{aligned} \tag{8}$$

Combining (6)–(8), we see that all Gabor transform values with $|\ell| > 2$ are asymptotically noise-dominated (with probability $\to 1$ as $N_s \to \infty$). Therefore, a hard thresholding, using threshold $T = 0.55\sigma N_s^{1/4}\sqrt{\log(N_s^{1/2})}$, is sufficient for large enough $N_s$ to remove all noise values (and signal transform values) with $|\ell| > 2$. [Note: Also all the $\mathcal{O}(1)$ errors for $|\ell| > 2$ in our first approximation are removed. Thus, at most, an $\mathcal{O}(1/N_s)$ contribution to MSE results from the remaining $\mathcal{O}(1)$ errors.]

Thus, the magnitude of signal error at $\tau_m$ due to losing Gabor transform values of the signal for $|\ell| > 2$ is bounded by (when performing an inverse FFT):

$$\frac{1}{\sqrt{N_s}} \sum_{|\ell|>2}^{\sqrt{N_s/2}} \mathcal{O}(\ell^{-2}) = \mathcal{O}(\frac{1}{\sqrt{N_s}}).$$

The MSE due to this loss is then bounded by

$$\frac{1}{N_s}\sum_{k=1}^{N_s} \mathcal{O}(\frac{1}{\sqrt{N_s}})^2 = \mathcal{O}(\frac{1}{N_s}).$$

For $|\ell| \leq 2$, we proceed as follows. Using (7), and assuming that *all* signal transform values are used [we just showed that neglecting them for $|\ell| > 2$ contributes only $\mathcal{O}(1/N_s)$ to MSE], we need only estimate for $|\ell| \leq 2$ the contribution to MSE of the noise values $\{\widetilde{\mathrm{n}}_k\}$ defined by

$$\widetilde{\mathrm{n}}_k = \frac{1}{\sqrt{N_s}} \sum_{\ell=-2}^{2} \mathcal{F}\{\mathrm{n}_k w_k\}[\ell] e^{i2\pi k\ell/\sqrt{N_s}}.$$

They all have standard deviation $\mathcal{O}(1/N_s^{1/4})$, because $\Re\mathcal{F}\{\mathrm{n}_k w_k\}$ and $\Im\mathcal{F}\{\mathrm{n}_k w_k\}$ have standard deviations of $0.55\sigma N_s^{1/4}/\sqrt{2}$. Hence their contribution to expected value is bounded by

$$\mathsf{E}(\frac{1}{N_s}\sum_k \widetilde{\mathrm{n}}_k^2) = \mathcal{O}(\frac{1}{\sqrt{N_s}}).$$

Finally, the perturbation due to the finite number of transform values ignored by considering only the $\mathcal{O}(\sqrt{N_s})$ values $t_k$ away from the any discontinuities of $g$, $g'$, and $g''$, contributes at most $\mathcal{O}[(\log N_s)^2/N_s]$ to MSE by similar considerations. *Q.E.D.*

*Remark 2:* (a) The proof reveals that the greatest contribution to MSE comes from the extremely low frequency data for $|\ell| \leq 2$. More research is needed to find a way to reduce MSE in that case. (b) Although Theorem 1 does not apply to the QuadChirp and MishMash data (which provide discrete models of distributions); nevertheless, their Gabor transforms are each converging to a small number of arcs in the time-frequency plane with values along the arcs growing at rate $\mathcal{O}(\sqrt{N_s})$. Hence, our proof can be modified by treating values of $\ell$ that are near the arcs (within $\pm 2$ for $\Delta\ell$) analogously to the way we treated $|\ell| \leq 2$, and treating values of $\ell$ away from the arcs (beyond $\pm 2$ for $\Delta\ell$) analogously to the way we treated $|\ell| > 2$. Thereby obtaining an expected value of MSE equal to $\mathcal{O}(1/\sqrt{N_s})$ for these two signal ensembles.

## Conclusion

We found that our new method for denoising one-dimensional signals, Hard Gabor (Blackman), outperforms the method in [5] and SureShrink for two-thirds of tested data, and outperforms BayesShrink on slightly more than half of tested data. The easy interpretation of the spectrograms is an added feature.

Future research will focus on several questions: (1) extending to "shrinkage" methods [6], [10]; (2) extending to non-stationary noise; (3) extending to non-uniformly sampled signals (our analysis in Section I applies to non-uniformly sampled signals); (4)

developing a model for separating noise from signal at extremely low frequencies (see Remark 2); (5) extending our convergence theorem to other signal data (such as discrete models of some class of distributions).

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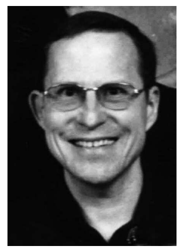

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